Concepts of Mathematics

October 19, 2005

Lecture 10

Corollary 1. *Let $z = r(\cos\theta + i\sin\theta)$. Then for any positive integer n,*

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Proof. For positive n it is easy to apply the De Moivre's rule. Note that

$$z^{-1} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{r}(\cos\theta - i\sin\theta) = r^{-1}\Big(\cos(-\theta) + i\sin(-\theta)\Big) = r_1(\cos\theta_1 + i\sin\theta_1),$$

where $r_1 = r^{-1}$ and $\theta_1 = -\theta$. Then for positive integer n,

$$z^{-n} = r_1^n(\cos n\theta_1 + i\sin n\theta_1) = r^{-n}\Big(\cos(-n\theta) + i\sin(-n\theta)\Big).$$

□

Definition 2. For any angle θ the complex number $\cos\theta + i\sin\theta$ is denoted by $e^{i\theta}$, i.e.,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Recall the trigonometric functions $\cos\theta$ and $\sin\theta$ are defined by

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}.$$

where $x^2 + y^2 = r^2$.

Theorem 3.

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}.$$

Example 1. Computer $(-1+\sqrt{3}i)^{20}$.

Let $\alpha = -1 + 2i$. Then $\alpha = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$. Thus

$$\alpha^{20} = 2^{20}\left(\cos\frac{40\pi}{3} + i\sin\frac{40\pi}{3}\right) = 2^{20}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 2^{19}(-1-\sqrt{3}i).$$

Example 2. Deriving trigonometric formulas. Consider $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$. Let $a = \cos\theta$, $b = \sin\theta$. Then

$$\begin{aligned}(a+bi)^3 &= (a^2 - b^2 + 2abi)(a+bi)\\ &= (a^2-b^2)a - 2ab^2 + (2a^2b + a^2b - b^3)i\\ &= a^3 - 3ab^2 + (3a^2b - b^3)i.\end{aligned}$$

Thus

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta.$$

Similarly,

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta.$$

Proposition 4. (a) *If $z = re^{i\theta}$, then $\bar{z} = re^{-i\theta}$.*

(b) *Let $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$. Then $z_1 = z_2$ if, and only if, $r_1 = r_2$ and $\theta_1 = \theta_2 + 2k\pi$ for some $k \in \mathbb{Z}$.*

Proof. (a) is obvious. (b) If $z_1 = z_2$, then $r_1 = r_2$, and $1 = z_1/z_2 = e^{i(\theta_1-\theta_2)}$. Hence $\theta_1 - \theta_2 = 2k\pi$ for some $k \in \mathbb{Z}$. The other part is obvious. □

1 Roots of unity

Definition 5. For any positive integer n, let $w = e^{\frac{2\pi}{n}}$; the nth roots of unity are the complex numbers

$$1, w, w^2, \ldots, w^{n-1}.$$

They are evenly distributed on the unit circle.

Example 3. For $n = 2$, $1, -1$; for $n = 4$, $1, i, -1, -i$; for $n = 3$,

$$1, \quad e^{\frac{2\pi i}{3}}, \quad e^{\frac{4\pi i}{3}}.$$

Theorem 6. *For any nth root of unity $w = e^{\frac{2\pi}{n}i}$ with $n \geq 2$,*

$$1 + w + w^2 + \cdots + w^{n-1} = 0.$$

Proof. Since $w^n = 1$ and $1 - w \neq 0$, then

$$(1-w)(1 + w + \cdots + w^{n-1}) = 1 - w^n = 0.$$

Hence $1 + w + \cdots + w^{n-1}$ must be zero. □

Lecture 11

2 Cubic Equations

The general cubic equation may be written as

$$x^3 + ax^2 + bx + c = 0. \tag{1}$$

Let $x = x - \frac{a}{3}$. Then $x^3 = (y - a/3)^3 = y^3 - ay^2 + (a^2/3)y - a^3/27$, $y^2 = x^2 - (2a/3)y + a^2/9$. Substitute $x = y - a/3$ into (1); the equation becomes the form

$$y^3 + 3hy + k = 0. \tag{2}$$

Let $y = u + v$. Then

$$y^3 = u^3 + v^3 + 3u^2v + 3uv^2 = u^3 + v^3 + 3uv(u+v) = u^3 + v^3 + 3uvy.$$

This means that the equation of the form $y^3 - 3uvy - (u^3 + v^3) = 0$ readily has a solution $y = u + v$. So we set

$$h = -uv, \quad k = -(u^3 + v^3).$$

Since $v = -h/u$, then $v^3 = -h^3/u^3$. Thus $k = -(u^3 - h^3/u^3)$ becomes

$$u^6 + ku^3 - h^3 = 0,$$

which is a quadratic equation in u^3. Then u^3 as

$$u^3 = \frac{-k + \sqrt{k^2 + 4h^3}}{2}.$$

Thus

$$v^3 = -k - u^3 = \frac{-k - \sqrt{k^2 + 4h^3}}{2}.$$

Therefore we obtain a solution

$$y = u + v = \sqrt[3]{\frac{-k + \sqrt{k^2 + 4h^3}}{2}} + \sqrt[3]{\frac{-k - \sqrt{k^2 + 4h^3}}{2}}.$$

There are three cubic roots for $u^3 = \frac{-k+\sqrt{k^2+4h^3}}{2}$ and also three cubic roots for $v^3 = \frac{-k-\sqrt{k^2+4h^3}}{2}$. So theoretically there are nine possible values to be the solutions; but there are only three solutions, some of them are the same.

Let u be a cubic root of $\frac{-k+\sqrt{k^2+4h^3}}{2}$, and let $\omega = e^{2\pi i/3}$. Then the other two cubic roots are $u\omega, u\omega^2$. Therefore the solutions for (2) are given by

$$u - \frac{h}{u}, \qquad u\omega - \frac{h\omega^2}{u}, \qquad u\omega^2 - \frac{h\omega}{u}.$$

Example 4. Consider the equation

$$x^3 - 3x + 2 = 0.$$

Since $h = -1$, $k = 2$, we have

$$u^3 = \frac{-k + \sqrt{k^2 + 4h^3}}{2} = -1.$$

So we have $u = -1$, thus the three solutions are given by

$$u - \frac{h}{u} = -2,$$

$$u\omega - \frac{h\omega^2}{u} = -\omega - \omega^2 = 1 - (1 + \omega + \omega^2) = 1,$$

$$u\omega^2 - \frac{h\omega}{u} = -\omega^2 - \omega = 1.$$

We may also solve the problem directly by the factorization $(x-1)(x-1)(x+2) = 0$.

Example 5. Consider the equation

$$x^3 - 6x - 6 = 0.$$

We have $h = -2$ and $k = -6$. Thus

$$u^3 = \frac{-k + \sqrt{k^2 + 4h^3}}{2} = 4.$$

So $u = \sqrt[3]{4}$. Thus

$$x_1 = u - \frac{h}{u} = 4^{1/3} + 2/4^{1/3} = 2^{2/3} + 2^{1/3},$$

$$x_2 = u\omega - \frac{h\omega^2}{u} = (2^{1/3} + 2^{2/3})\omega + (2^{-1/3} + 2^{-2/3})^{-1}\omega^2,$$

$$x_3 = u\omega^2 - \frac{h\omega}{u} = (2^{-1/3} + 2^{-2/3})^{-1}\omega + (2^{1/3} + 2^{2/3})\omega^2.$$

3 Fundamental Theorem of Algebra

Theorem 7. *Every polynomial equation of degree at leat 1 has a root in $\mathbb{C}$.*

Theorem 8. *Every polynomial of degree n factories as a product of linear polynomials, and has exactly n roots (counted with multiplicity) in $\mathbb{C}$.*

Proposition 9. *Let $\alpha_1, \ldots, \alpha_n$ the roots of the equation*

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Then

$$s_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_n = -a_{n-1}$$

$$s_2 = \sum_{i<j} \alpha_i \alpha_j = a_{n-2},$$

$$s_3 = \sum_{i<j<k} \alpha_i \alpha_j \alpha_k = a_{n-3},$$

$$\cdots,$$

$$s_n = \alpha_1 \alpha_2 \cdots \alpha_n = (-1)^n a_0.$$

Example 6. Find a cubic equation with roots $2 + i$, $2 - i$, and 3.

$$s_1 = \alpha_1 + \alpha_2 + \alpha_3 = 7,$$

$$s_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = 17,$$

$$s_3 = \alpha_1\alpha_2\alpha_3 = 15.$$

$$(x - 3)(x^2 - 4x + 5) = x^3 - 7x^2 + 17x - 15 = 0.$$

Example 7. Let α and β be roots of equation $x^2 - 5x + 9 = 0$. Find a quadratic equation with roots α^2 and β^2.

The quadratic equation is of the form

$$x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0.$$

Since $\alpha + \beta = 5$ and $\alpha\beta = 9$, we have $5^2 = (\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta = \alpha^2 + \beta^2 + 18$. Then $\alpha^2 + \beta^2 = 7$, $\alpha^2\beta^2 = 81$. Thus

$$x^2 - 7x + 81 = 0.$$